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Periodic and Solitary Wave Solutions of the Three-Wave Problem. A Different Approach

A. Roy Chowdhury[1], M. Debnath[1], and R. S. Banerjee[1]

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We consider the inverse problem for a three-wave interaction system in a manner different from Zakharov *et al.* and of Kaup. Our method is an adaptation of the technique due to Date to a 3×3 Lax pair. The analysis leads to a system of ordinary nonlinear equations for the μ_i variables linearizable through a suitable definition of differential on a Riemann surface. Next, in the degenerate case, when the μ_i are equal in pairs, we prove that such a set of equations is exactly integrable and leads to solitary solutions.

1. FORMULATION

The equations describing the dispersion three-wave process are

$$
\begin{aligned}
Q_{1t}+c_1Q_{1x}&=ir_1Q_2^*Q_3^*\\
Q_{2t}+c_2Q_{2x}&=ir_2Q_1^*Q_3^*\\
Q_{3t}+c_3Q_{3x}&=ir_3Q_2^*Q_1^*
\end{aligned}
\tag{1}
$$

This set of equations has been discussed in detail by Zakharov *et al.* (1984) and Kaup (1972) from the viewpoint of inverse scattering. We will also start from the Lax pair utilized in Zakharov *et al.* (1984). The x part of the Lax equation is written as

$$
\psi_x=\begin{pmatrix} i\lambda d_1 & Q_1'(x) & Q_2'(x)\\ a_1Q_1^{*\prime}(x) & i\lambda d_2 & Q_3'(x)\\ a_2Q_2^{*\prime}(x) & a_3Q_3^{*\prime}(x) & i\lambda d_3\end{pmatrix}\psi \tag{2}
$$

[1]High Energy Physics Division, Physics Department, Jadavpur University, Calcutta 700 032, India.

or

$$\psi_x = L\psi$$

2. INTRODUCTION

The inverse problem for the three-wave process has been dealt with in detail by Kaup (1972), and Zakharov *et al.* (1984). The case they considered was the usual asymptotically decaying case. On the other hand, another case of equal importance is the periodic inverse problem, which was solved for the cases of nonlinear Schrödinger (Roy Chowdhury *et al.*, 1985) and its variants, the sine-Gordon (Forest and Maclaughlin, 1982) and Thirring models (Date, 1978), essentially following the idea of Date and Tanaka (1976). In this paper we extend this philosophy to a 3×3 inverse problem and show that it is possible to solve both the asymptotic and periodic case at one.

In the following we will also require the adjoint of the Lax equation. Let us denote the adjoint wavefunction as $\bar{\psi}$ and the corresponding matrix as $\bar{L}$,

$$\bar{\psi}_x = \bar{\psi}\bar{L} \tag{3}$$

From equations (2) and (3) we easily deduce

$$(f_{11})_x = Q_1' f_{21} + Q_2' f_{31} - a_1 Q_1^{*\prime} f_{12} - a_2 Q_2^{*\prime} f_{13} \tag{4a}$$

$$(f_{22})_x = a_1 Q_1^{*\prime} f_{12} + Q_3' f_{32} - Q_1' f_{21} - a_3 Q_3^{*\prime} f_{23} \tag{4b}$$

$$(f_{33})_x = a_2 Q_2^{*\prime} f_{13} + a_3 Q_3^{*\prime} f_{23} - Q_2' f_{31} - Q_3' f_{32} \tag{4c}$$

$$(f_{12})_x = i\lambda(d_1 - d_2) f_{12} + Q_1'(f_{22}) + Q_2'(f_{32}) - Q_1'(f_{11}) - a_3 Q_3^{*\prime}(f_{13}) \tag{5a}$$

$$(f_{21})_x = i\lambda(d_2 - d_1) f_{21} + a_1 Q_1^{*\prime}(f_{11}) + Q_3'(f_{31}) - a_1 Q_1^{*\prime} f_{22} - a_2 Q_2^{*\prime} f_{23} \tag{5b}$$

$$(f_{13})_x = i\lambda(d_1 - d_3) f_{13} + Q_1'(f_{23}) + Q_2' f_{33} - Q_2' f_{11} - Q_3' f_{12} \tag{5c}$$

$$(f_{31})_x = i\lambda(d_3 - d_1) f_{31} + a_2 Q_2^{*\prime}(f_{11}) + a_3 Q_3^{*\prime} f_{21} - a_1 Q_1^{*\prime} f_{32} - a_2 Q_2^{*\prime} f_{33} \tag{5d}$$

and similarly

$$(f_{23})_x = i\lambda(d_2 - d_3) f_{23} + a_1 Q_1^{*\prime} f_{13} + Q_3' f_{33} - Q_2' f_{21} - Q_3' f_{22} \tag{5e}$$

$$(f_{32})_x = i\lambda(d_3 - d_2) f_{32} + a_2 Q_2^{*\prime} f_{12} + a_3 Q_3^{*\prime} f_{22} - Q_1' f_{31} - a_3 Q_3^{*\prime} f_{33} \tag{5f}$$

where the f_{ij} are the following product eigenfunctions:

$$f_{ij} = \psi_i \bar{\psi}_j \qquad (i, j = 1, 2, 3)$$

We then compute and observe that the following equations hold:

$$\begin{aligned} (f_{12} f_{21} - f_{11} f_{22})_{x,t} &= 0 \\ (f_{13} f_{31} - f_{11} f_{33})_{x,t} &= 0 \\ (f_{23} f_{32} - f_{22} f_{33})_{x,t} &= 0 \end{aligned} \tag{6}$$

where the subscripts x, t denote differentiation with respect to the corresponding variable. Actually, an identical set of equations is obtained for the temporal evolution of f_{ij} and so we have stated both the (x, t) parts simultaneously. Equations (6) immediately imply that the quantities inside the brackets are constants both in space and time, and the eigenfunctions themselves being analytic in λ are analytic functions of λ. Hence we set

$$f_{ii}=\sum_0^{N+1} f_{ii}^k\lambda^{2k}$$
$$f_{ij}=\sum_0^{N} f_{ij}^k\lambda^{2k+1}, \qquad i\neq j \tag{7}$$

Substituting in (5a)-(5d) etc., and comparing various powers of λ, we get

$$Q_1'=\frac{i(d_2-d_1)}{f_{22}^{N+1}-f_{11}^{N+1}}f_{12}^N \tag{8}$$

along with

$$Q_1^{*\prime}=\frac{i(d_2-d_1)f_{21}^N}{a_1(f_{22}^{N+1}-f_{11}^{N+1})} \tag{9}$$

Equations (8) and (9) immediately suggest

$$f_{12}^{*N}=-f_{21}^N/a_1 \tag{10}$$

Similar results hold for Q_2, Q_3, and other f_{ij} $(i\neq j)$. For example,

$$Q_2'=\frac{i(d_3-d_1)f_{13}^N}{f_{33}^{N+1}-f_{11}^{N+1}}; \qquad Q_3'=\frac{i(d_3-d_2)f_{23}^N}{f_{33}^{N+1}-f_{11}^{N+1}} \tag{11}$$

These equations are useful for the inverse problem of determining the nonlinear field variables in terms of the information available for f_{ij}^k. In the following we elaborate on this aspect.

To proceed further, we now write the integrals of motion of equations (6) as

$$(f_{12}f_{21}-f_{11}f_{22})=P(\lambda^2)=\sum_0^{2N+2} P_k\lambda^{2k}$$
$$(f_{13}f_{31}-f_{11}f_{33})=Q(\lambda^2)=\sum_0^{2N+2} Q_k\lambda^{2k} \tag{12}$$
$$(f_{23}f_{32}-f_{22}f_{33})=Q(\lambda^2)=\sum_0^{2N+2} R_k\lambda^{2k}$$

In further analogy with Forest and Maclaughlin (1982) and Date and Tanaka (1976), specify the zeros of the analytic functions f_{11}, f_{22} and f_{33} to be at the positions as follows:

$$f_{11}=f_{11}^{N+1}\prod_{j=1}^{N+1}[\lambda^2-\mu_j(x,t)]$$
$$f_{22}=f_{22}^{N+1}\prod_{j=1}^{N+1}[\lambda^2-\nu_j(x,t)] \tag{13}$$
$$f_{33}=f_{33}^{N+1}\prod_{j=1}^{N+1}[\lambda^2-\sigma_j(x,t)]$$

2.1. Equations Describing the (μ, ν, σ) Dynamics

Before proceeding further, we rewrite the polynomials P, Q, and R as finite products in the form

$$P(\lambda^2)=\prod_{j=0}^{2N+1}(\lambda^2-A_j)$$

with similar expressions for Q and R.

Now substituting (13) in (4a)-(4c) and proceeding to the respective zeros $\lambda^2=\mu_j, \partial_j$ or ν_j, we obtain

$$\frac{\partial \nu_j}{\partial x}=-i(d_1-d_2)\nu_j^{1/2}\frac{f_{12}^N-f_{21}^N}{f_{22}^{N+1}(f_{22}^{N+1}-f_{11}^{N+1})} \times L^{-1}\prod_{j=1}^{N+1}(\nu_j-A_j)+i(d_2-d_3)\nu_j^{1/2}\frac{f_{23}^N-f_{32}^N}{f_{22}^{N+1}(f_{22}^{N+1}-f_{11}^{N+1})} \times L^{-1}\prod_{j=1}^{N+1}(\nu_j-C_j) \tag{14}$$

$$\frac{\partial \sigma_j}{\partial x}=6_j^{1/2}[f_{33}^{N+1}M(f_{33}^{N+1}-f_{11}^{N+1})]^{-1} \times\left[-i(d_1-d_3)\prod_{i=1}^{N+1}(6_j-B_j)(f_{13}^N-f_{31}^N)-i(d_2-d_3) \times\prod_{i=1}^{N+1}(6_j-C_j)(f_{23}^N-f_{23}^N)\right] \tag{15}$$

$$\frac{\partial \mu_j}{\partial x}=\mu_j^{1/2}[f_{11}^{N+1}N(f_{22}^{N+1}-f_{11}^{N+1})]^{-1} \times\left[i(d_1-d_2)\prod_{j=1}^{N+1}(\mu_j-A_j)(f_{12}^N-f_{21}^N) +i(d_1-d_3)\prod_{j=1}^{N+1}(\mu_j-B_j)(f_{13}^N-f_{31}^N)\right] \tag{16}$$

where N, M, L are the roots of the equation

$$(\mu^{\nu}-A_1)(\mu^{\nu}-A_2)(\mu^{\nu}-A_3)=0.$$

We now go back to the definition of P, Q, and R, and substitute the expressions for f_{ij} in them, and equate various powers of λ, which leads to the following useful identities:

$$\begin{aligned} f_{11}^{N+1}f_{22}^{N+1} &= -P_{2N+2} \\ f_{12}^{N}f_{21}^{N}+f_{11}^{N+1}f_{22}^{N+1}\left(\sum \mu_j+\sum \nu_j\right) &= P_{2N+1} \\ f_{11}^{N+1}f_{33}^{N+1} &= -Q_{2N+2} \\ f_{13}^{N}f_{31}^{N}-f_{11}^{N+1}f_{33}^{N+1}\left(\sum \mu_j+\sum \sigma_j\right) &= Q_{2N+1} \end{aligned} \tag{17}$$

Utilizing these, we can immediately write equations (14) in the form

$$\begin{aligned} \frac{\partial \nu_j}{\partial x} &= (d_1-d_2)\nu_j^{1/2}[L(P_{2N+2}+(f_{22}^{N+1})^2]^{-1} \\ &\quad\times\left\{(1+a_1)\left[P_{2N+1}+P_{2N+2}\left(\sum \mu_j+\sum \nu_j\right)\right]^{1/2}\prod_{j=1}^{N+1}(\nu_j-A_j)\right\} \\ &\quad+(d_2-d_3)\nu_j^{1/2}[L(R_{2N+2}+(f_{22}^{N+1})^2]^{-1} \\ &\quad\times\left\{(1+a_3)\left[R_{2N+1}+R_{2N+2}\left(\sum \nu_j+\sum \sigma_j\right)\right]^{1/2}\prod_{j=1}^{N+1}(\nu_j-C_j)\right\} \end{aligned} \tag{18}$$

Equation (18) contains only the variables (μ, ν, σ); all others are constants. So equation (18) with two other similar equations for σ and μ form a coupled set of ordinary nonlinear differential equations for (μ, ν, σ). On the other hand, an identical set of equations can also be deduced for their evolution with time. Note that in the above discussions we never assumed that the nonlinear fields Q_1, Q_2, Q_3 vanish as $x\to\pm\infty$. The next step is to define suitable differentials on a properly defined Riemann surface which will linearize the flow of x and t and with the help of Abel's inversion theorem enable us to determine (μ, ν, σ) in terms of θ functions. But this last step is not so instructive, since it is of an abstract nature, so we do not pursue it any more, but show a different way which will lead to an explicit analysis of these equations governing the dynamics of (μ, ν, σ).

2.2. Reduction of Equations

In this section we start with the assumption that the zeros μ_j, ν_j, and σ_j are pairwise equal; in that case these equations [i.e., (15), (16), or (18)] can be reduced to the form

$$\frac{1}{\mu_j^{1/2}}\frac{1}{(a+b\sum \mu_j)^{1/2}}\frac{\partial \mu_j}{\partial x}=\frac{\alpha\prod_{j=1}^{N+1}(\mu_j-A_j)}{\prod_{i\neq j}(\mu_j-\mu_i)}+\frac{\beta\prod_{j=1}^{N+1}(\mu_j-B_j)}{\prod_{i\neq j}(\mu_i-\mu_j)}$$

Instead of considering the general case, we consider only $N=2$, whence we are led to

$$\frac{1}{\mu_1^{1/2}}\mu_{1x}\frac{1}{f(\mu_1)}+\frac{1}{\mu_2^{1/2}}\mu_{2x}\frac{1}{f(\mu_2)}+\frac{1}{\mu_3^{1/2}}\mu_{3x}\frac{1}{f(\mu_3)}=0$$

$$\mu_1^{1/2}\mu_{1x}\frac{1}{f(\mu_1)}+\mu_2^{1/2}\mu_{2x}\frac{1}{f(\mu_2)}+\mu_3^{1/2}\mu_{3x}\frac{1}{f(\mu_3)}=0$$

$$\mu_1^{3/2}\mu_{1x}\frac{1}{f(\mu_1)}+\mu_2^{3/2}\mu_{2x}\frac{1}{f(\mu_2)}+\mu_3^{3/2}\mu_{3x}\frac{1}{f(\mu_3)}=0 \tag{19}$$

where

$$f(\mu)=\alpha(\mu-A_1)(\mu-A_2)(\mu-A_3)+\beta(\mu-B_1)(\mu-B_2)(\mu-B_3)$$
$$=\text{a cubic polynomial in } \mu$$

Our primary aim is to solve for μ_1, μ_2, μ_3 from the three equations in (19), but before that, these are to be integrated.

The first equation of (19) after quadrature leads to

$$\frac{\prod_{i=1}^{3}(\mu_i^{1/2}-L^{1/2})^A(\mu_i^{1/2}-M^{1/2})^B(\mu_i^{1/2}-N^{1/2})^C}{\prod_{i=1}^{3}(\mu_i^{1/2}+L^{1/2})^A(\mu_i^{1/2}+M^{1/2})^B(\mu_i^{1/2}+N^{1/2})^C}=e^{k_1} \tag{20}$$

whence the second and third equations of (19) yield respectively,

$$(2L^{1/2}A+2M^{1/2}B+2N^{1/2}C)\sum_{i=1}^{3}\mu_i^{1/2}+\log\frac{\tilde{N}}{\tilde{D}}=k_2 \tag{21}$$

$$\tilde{N}=\prod_{i=1}^{3}(\mu_i^{1/2}-L^{1/2})^{LA}(\mu_i^{1/2}-M^{1/2})^{MB}(\mu_i^{1/2}-N^{1/2})^{NC}$$

$$\tilde{D}=\prod_{i=1}^{3}(\mu_i^{1/2}+L^{1/2})^{LA}(\mu_i^{1/2}-M^{1/2})^{MB}(\mu_i^{1/2}-N^{1/2})^{NC}$$

and

$$(2L^{3/2}A+2M^{3/2}B+2N^{3/2}C)\sum_{i=1}^{3}i+(2LA^{1/2}+2MB^{1/2}$$
$$+2NC^{1/2})\tfrac{1}{3}\sum\mu_i^{3/2}+\log\frac{\tilde{\tilde{N}}}{\tilde{\tilde{D}}}=x+k_3 \tag{22}$$

$$\tilde{\tilde{N}}=\prod_{i=1}^{3}(\mu_i^{1/2}=L^{1/2})^{L^2A}(\mu_i^{1/2}-M^{1/2})^{M^2B}(\mu_i^{1/2}-N^{1/2})^{N^2C}$$

$$\tilde{\tilde{D}}=\prod_{i=1}^{3}(\mu_i^{1/2}+L^{1/2})^{L^2A}(\mu_i^{1/2}+M^{1/2})^{M^2B}(\mu_i^{1/2}+N^{1/2})^{N^2C}$$

So equations (20)-(22) are now algebraic equations to be solved for μ_i.

2.3. Special case

By a suitable choice we can make the constants L, M, N all equal and in this situation we arrive at the following equations:

$$\frac{1}{8}\sum_{i=1}^{3}\frac{3\mu_i-5L}{L^2(\mu_1-L)^2}\mu_i^{1/2}+\ln\left[\frac{\prod_{i=1}^{3}(\mu_i^{1/2}-L^{1/2})^A}{\prod_{i=1}^{3}(\mu_i+L^{1/2})^A}\right]=k_1 \tag{23}$$

$$+\frac{1}{8}\sum_{i=1}^{3}\frac{\mu_i L}{L^2(\mu_i-L)^2}\mu_i^{1/2}+\ln\left[\frac{\prod_{i=1}^{3}(\mu_i^{1/2}-L^{1/2})^{AL/3}}{\prod_{i=1}^{3}(\mu_i^{1/2}+L^{1/2})^{AL/3}}\right]=k_2 \tag{24}$$

$$=\frac{1}{8}\sum_{i=1}^{3}\frac{\mu_i+L}{L^2(\mu_i-L)^2}\mu_i^{5/2}+\ln\left[\frac{\prod_{i=1}^{3}(\mu_i^{1/2}-L^{1/2})^{AL^2}}{\prod_{i=1}^{3}(\mu_i^{1/2}+L^{1/2})^{AL^2}}\right]$$

$$+\frac{1}{16L^2}\sum(\mu_i+6L)\mu_i^{3/2}=\lambda x+k_3 \tag{25}$$

which are to be solved for μ_i. Similar equations can be deduced for σ_i and ν_i. When all these are known, then we know f_{ij}—the analogue of square eigenfunctions, whence the inverse problem is solved via equations (8) and (9).

3. DISCUSSIONS

In the above analysis we have presented a method for the periodic inverse problem of the three-wave problem. Our method is essentially an extension of the procedure due to Date. Furthermore, from the degeneracy condition of the zeros of the squared eigenfunctions we can deduce some simpler equation for (μ_i, ν_i, σ_i) which are algebraic in nature and can lead to the solitary wave solutions. So they actually combine both classes of solutions together.

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